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SOME MORE PRIMITIVE GROUP RINGS

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ABSTRACT

In this paper, we show that the group rings of several families of groups are primitive. Let A and B be two groups with $1 < |A| \leqq |B|$ and B infinite. Then the main result is that if K is a field for which $K[A^\omega]$ is semiprimitive, then $K[A \wr B]$ is primitive. In addition, the field may be replaced by a subdomain in case A is not torsion or A is not locally finite and K has characteristic 0. Certain other wreath products and free products are discussed.

1. Introduction

The primary purpose of this paper is to prove that the group rings of a large family of wreath products are primitive. Let A and B be two groups, and denote by A^B the restricted direct product of $|B|$-many copies of A, indexed by B. The wreath product $A \wr B$ is defined to be the semi-direct product of A^B by B, with respect to the action

$$ba_{b'}b^{-1} = a_{bb'}.$$

Denote by A^λ the restricted direct product of λ-many copies of A, for any cardinal λ, and let ω be the first infinite cardinal. Recall that a ring is called semiprimitive if its Jacobson radical is (0). Our main results deal with a group A and an infinite group B such that $|B| \geqq |A| > 1$. The basic theorem is

THEOREM 1.1. *Let K be a field such that $K[A^\omega]$ is semiprimitive. Then $K[A \wr B]$ is primitive.*

More generally, we will prove

THEOREM 1.2. *Let S be a subring of K such that K is generated as S-algebra by $\leqq |B|$ elements. Then $S[A \wr B]$ is primitive in case* (i) *A is not torsion or* (ii) *A is not locally finite and K has characteristic* 0.

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As a corollary, we will obtain

THEOREM 1.3. *Assume that F is a field of characteristic 0, or that F has characteristic p and A has an element of infinite order, but no elements of order p. Then $F[A^{\omega}]$ is semiprimitive.*

This result is only new in case F is algebraic over its prime subfield. It applies in turn to 1.1 and 1.2, showing that they hold for any field of characteristic 0.

Conditions (i) and (ii) of 1.2 are precisely those which are known to imply that $K[A]$ is not algebraic. The problem of proving primitivity of $S[A \setminus B]$ for S a domain of characteristic p and A torsion but not locally finite seems closely related to that of deciding whether $K[A]$ is algebraic in this case, an old problem of Herstein. A more general question of his is whether the condition that every regular element in a group ring is invertible forces the group to be locally finite. Formanek showed that this is the case in characteristic 0, and an extension of his main argument to real-closed fields is what allows us to prove case (ii) of 1.3. We note that as one consequence, there exist finitely-generated torsion groups whose group rings are primitive over any field or countable domain of characteristic 0.

These results substantially generalize an earlier theorem that $R[\mathbb{Z} \setminus B]$ is primitive for any infinite group B and R a field or countable domain [5]. The proofs are given in section 3, while some preliminaries which are needed are collected in section 2.

In the case that no semiprimitivity assumption is available, Passman has proved [10, 9.2.8]:

THEOREM 1.4. *Let P be a finite p-group and K a field of characteristic p. Then $K[P \setminus \mathbb{Z}]$ is primitive.*

In section 4, we describe the construction of explicit faithful, simple modules for a family of group rings including these. Another special case arises in section 5, where we prove

THEOREM 1.5. *Let G be the group generated by x and y with the relation $x^{-1}yx = y^r$ for $r > 1$. Then $F[G]$ is primitive for any field F.*

In case F is algebraically closed and not absolute, the proof is particularly easy (see 5.1), and probably provides as simple an example of primitive group rings as one could desire.

Finally, we take a brief look at free products. Formanek proved that for two groups A and B, with $|A| > 2$, the group ring $R[A * B]$ is primitive for R a field or a domain with $|R| \leqq |A * B|$ [3; 10, 9.2.10]. We showed in [5] how one can construct explicit modules in case A is infinite and an additional cardinality assumption is made. In section 6, we exhibit a construction of a faithful, simple module for A finite and B cyclic of finite order, which can be extended to other free products.

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2. Preliminaries

We collect here a number of well-known results on algebras and group rings and some additional results, all of which are used in the next section. In studying simple modules, it is convenient to work over a large algebraically closed field, as the following result suggests. We denote by ω the first infinite cardinal, as before.

THEOREM 2.1. *Let R be an algebra over an algebraically closed field K such that $(\omega)(\dim_K R) < |K|$. Then the endomorphism ring of any simple R-module is K.*

The case that R is finitely-generated and K is uncountable is well-known, but the proof of the general result is the same [2]. This is useful in dealing with tensor products, in combination with a classical result of Azumaya and Nakayama [6, p. 114]:

THEOREM 2.2. *Let R, S be two K-algebras with simple modules M and N respectively. If $\mathrm{End}_R M = K$, then $M \bigotimes_K N$ is a simple $R \bigotimes_K S$-module.*

Tensor products of group algebras are a natural construction, in view of the next result [10, 1.3.4, 7.3.10–12]:

THEOREM 2.3. *Let A and B be two groups and K a field. Then*

$$K[A] \bigotimes_K K[B] = K[A \times B]$$

and $K[A]$ is separable if it is semiprimitive. Hence $K[A \times B]$ is semiprimitive if $K[A]$ and $K[B]$ are.

Our strategy in proving the wreath product results is to work over a large field so that we can use the above theorems, and then descend to smaller fields. The second step uses an intersection theorem of M. Smith [10, 9.1.3]:

THEOREM 2.4. *Let $F \subset K$ be a field extension and let G be a group whose finite conjugate subgroup $\Delta(G) = 1$. Then any non-zero ideal of $K[G]$ has non-zero intersection with $F[G]$.*

Given groups A and B with B infinite, the wreath product $A \setminus B$ has trivial finite conjugate subgroup, so that 2.4 will apply. Another intersection theorem allows us to ignore B in proving that certain $K[A \setminus B]$-modules are faithful [10, 9.2.7]:

THEOREM 2.5. *Let $A \neq 1$ and let B be infinite. Any non-zero ideal of $K[A \setminus B]$ has non-zero intersection with $K[A^B]$.*

One more intersection theorem we use is [10, 7.2.10]:

PROPOSITION 2.6. *Let H be a normal subgroup of G for which G/H is locally finite. If $K[G]$ is semiprimitive, so is $K[H]$.*

The next collection of results allows us to descend primitivity to certain smaller base rings. We begin with an easy fact:

PROPOSITION 2.7. *Let a be an element of infinite order in a group A and let c be a non-zero element of a field K. Then $a - c$ is not a unit in $K[A]$.*

PROOF. Suppose there exist elements $c_i \in K$ and $a_i \in A$ such that

$$(a-c)\sum_{i=1}^{m} c_i a_i = 1.$$

Then

$$c_1 a a_1 + \cdots + c_m a a_m - c c_1 a_1 - \cdots - c c_m a_m = 1.$$

This means each a_i equals $a a_j$ for some j, and $a_i = a^k a_i$ for some i and k. Thus a has finite order, a contradiction.

The following extension is essentially in Formanek's paper [4], or [10, 2.3.13].

PROPOSITION 2.8. *Let A be an infinite group generated by $x_1, \cdots, x_n$. Let $c \geqq n+1$ be an element of $\mathbb{R}$. Then $x_1 + \cdots + x_n - c$ is not a unit in $\mathbb{R}[A]$.*

PROOF. Let $\alpha = (1/c)(x_1 + \cdots + x_n)$. We want to prove that $\alpha - 1$ is not a unit. Suppose it is, with β as its inverse. We will obtain a contradiction by proving that $A \subset \mathrm{Supp}(\beta)$. Let

$$\gamma_i = (1 + \alpha + \cdots + \alpha^i) - \beta.$$

Then $\gamma_i(1-\alpha) = -\alpha^{i+1}$, so that $\gamma_i = -\alpha^{i+1}\beta$. Define the norm

$$\left|\sum c_i a_i\right| = \sum |c_i|.$$

Then

$$|\gamma_i| = |-\alpha^{i+1}\beta| \leqq |\alpha|^{i+1}|\beta| = \left(\frac{n}{c}\right)^{i+1}|\beta|.$$

Since $c > n$, we see that $\lim_{i\to\infty}|\gamma_i| = 0$.

Let $a = x_{i_1}\cdots x_{i_m}$ be an element of A, and let c_j be its coefficient in α^j. We wish to show that $a \in \mathrm{Supp}(\beta)$. If not, then the coefficient of a in γ_i must be $c_0 + \cdots + c_i$, and for $i \geqq m$ we find that

$$|\gamma_i| \geqq c_0 + \cdots + c_i \geqq c_m > 0.$$

This contradicts the fact that $|\gamma_i|$ approaches 0.

We will need the following extension of 2.8:

THEOREM 2.9. *Let A be an infinite group generated by $x_1, \cdots, x_n$. Let K be a real-closed field with $c \geqq n+1$ an element of K. Then $x_1 + \cdots + x_n - c$ is not a unit in $K[A]$.*

PROOF. Suppose $x_1 + \cdots + x_n - c$ is a unit, with inverse $\sum_{j=1}^m c_j a_j$. Then $c, c_1, \cdots, c_m$ provide a solution in K to a system of equations in variables $T, T_1, \cdots, T_m$ over $\mathbb{Q}$, along with the inequality $T \geqq n+1$. By Tarski's result that the theory of real-closed fields is complete [7, p. 312], there must also be a solution $r, r_1, \cdots, r_m$ in $\mathbb{R}$, providing an inverse to $x_1 + \cdots + x_n - r$. Since $r \geqq n+1$, this contradicts 2.8.

An analogue of 2.9 in characteristic p would allow us to extend case (ii) Theorem 1.2 to fields of characteristic p, and would answer the questions of Herstein mentioned in the introduction.

The primitivity proofs of section 3 involve the construction of modules via an infinite tensor product. We introduce the needed construction here. Let A be a group, β and λ two cardinal numbers, and K a field. Suppose for each $\alpha \in \beta$ we have a $K[A^\lambda]$-module V_α on which only a finite number of copies of A act non-trivially, so that any single A acts non-trivially on at most one V_α. In each V_α, choose a vector $v_\alpha \neq 0$. We wish to define a new $K[A^\lambda]$-module, which we think of as the pointed tensor product of the pointed spaces (V_α, v_α). First, let $\bar{V}$ be the K-vector space whose basis is the set of "tensors"

$$\left\{\bigotimes_{\alpha\in\beta} w_\alpha : w_\alpha = v_\alpha \text{ for all but finitely many } \alpha\right\}.$$

Let $W \subset \bar{V}$ be the subspace of multilinear relations. For instance, for $c \in K$, the space W contains

$$cw_\gamma \otimes \bigotimes_{\alpha \neq \gamma} w_\alpha - c(\otimes w_\alpha), \qquad \text{and}$$

$$(w_\gamma^1 + w_\gamma^2) \otimes \bigotimes_{\alpha \neq \gamma} w_\alpha - w_\gamma^1 \otimes \bigotimes_{\alpha \neq \gamma} w - w_\gamma^2 \otimes \bigotimes_{\alpha \neq \gamma} w_\alpha .$$

Then we define the space $V = \bigotimes_{\alpha \in \beta} (V_\alpha, v_\alpha)$ to be $\bar{V}/W$. It is a $K[A^\lambda]$-module in a natural way; given $a \in A^\lambda$, we let

$$(\otimes w_\alpha) \cdot a = \otimes (w_\alpha a).$$

Since a acts non-trivially on only finitely many entries, this is well-defined. The action is extended linearly to $K[A^\lambda]$. We use the following result:

THEOREM 2.10. *Let (V_α, v_α) be a collection of pointed modules over $K[A^\lambda]$. Then the space $V = \bigotimes (V_\alpha, v_\alpha)$ is non-zero. If, in addition, each V_α is simple and K is an algebraically closed field with $|K| > |A| \omega$, then V is simple.*

PROOF. We claim that the image of $\bigotimes v_\alpha$ in V is non-zero. Otherwise $\bigotimes v_\alpha$ lies in W. Suppose $\bigotimes v_\alpha = \Sigma w_i$, and the finite collection of entries where some w_i differs from the distinguished vector is $\alpha_1, \cdots, \alpha_n$. Then we find that

$$v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_n}$$

lies in the submodule of relations which define the tensor product $V_{\alpha_1} \otimes \cdots \otimes V_{\alpha_n}$. Thus $v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_n} = 0$, which is absurd.

Turning to the second statement, we must show that any $v \neq 0$ in V is cyclic. The basis vectors involved in v differ from $\bigotimes v_\alpha$ in only finitely many entries. Let us again denote these entries by $\alpha_1, \cdots, \alpha_n$, and let $A_{\gamma_1}, \cdots, A_{\gamma_m}$ be the finitely many copies of A which do not act trivially on $V_{\alpha_1}, \cdots, V_{\alpha_n}$. Then we may decompose V as $V_1 \otimes V_2$, where

$$V_1 = V_{\alpha_1} \otimes \cdots \otimes V_{\alpha_n}, \qquad V_2 = \bigotimes_{\alpha \neq \alpha_i} (V_\alpha, v_\alpha).$$

The vector $v = v_1 \otimes v_2$ for some $v_1 \neq 0$ in V_1 and $v_2 = \bigotimes v_\alpha$. The $K[A^\lambda]$ structure on V_1 is essentially that of $R = K[A_{\gamma_1} \times \cdots \times A_{\gamma_m}]$, so by the assumption on K, we can apply 2.1 and 2.2 to deduce that V_1 is simple over R. Thus for some $r \in R$,

$$v_1 \cdot r = v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_n},$$

and since each group element in r fixes v_2, we find by multilinearity that

$$v \cdot r = v_1 r \otimes v_2 = \bigotimes v_\alpha.$$

This vector is obviously cyclic, so v is as well, and V is simple.

3. The wreath product results

We begin by proving

THEOREM 3.1. *Let A and B be two groups with B infinite and $|B| \geqq |A| > 1$. Let K be a field such that $K[A^\omega]$ is semiprimitive. Then $K[A \wr B]$ is primitive.*

PROOF. Let F be the prime subfield of K and let β be the least ordinal with $|\beta| = |B|$. Fix a 1–1 correspondence between β and $F[A^B]$, assigning p_α to the ordinal α. We will define inductively a family of $K[A^B]$-modules V_α so that only finitely many copies of A act non-trivially on V_α, and each A acts non-trivially on at most one V_α.

Write A_b for the copy of A in A^B indexed by $b \in B$, and let

$$Y_\alpha = \{b \in B : \text{the action of } A_b \text{ is defined non-trivially on } V_\gamma \text{ for some } \gamma < \alpha\}.$$

For any $\alpha < \beta$, we have $|Y_\alpha| < \beta$, so that at any stage α there are still β-many copies of A for which a non-trivial action has not yet been assigned.

We construct V_α as follows. The element p_α may be viewed as lying in the group ring of a finite number of copies of A. Identifying these with some copies of A in another algebra $K[A^\omega]$, we obtain an element $p \in K[A^\omega]$ corresponding to p_α. Since $K[A^\omega]$ is semiprimitive, it contains an element q for which $1 + pq$ is not a unit. Let q_α in $K[A^B]$ correspond to q, so that $1 + p_\alpha q_\alpha$ is not a unit, and assume that $p_\alpha q_\alpha$ lies in $K[A_{b_1} \times \cdots \times A_{b_m}]$. Then choose b_α so that no $b_\alpha b_i$ lies in Y_α. This is possible, for otherwise Y_α must contain β many elements.

The element $1 + b_\alpha p_\alpha q_\alpha b_\alpha^{-1}$ is a non-unit in $K[A_{b_\alpha b_1} \times \cdots \times A_{b_\alpha b_m}]$ so there is a maximal right ideal m_α of this ring containing $1 + b_\alpha p_\alpha q_\alpha b_\alpha^{-1}$. We define V_α to be the corresponding simple module, with any other copy of A acting trivially, and choose $v_\alpha \in V_\alpha$ to be the image of 1. Thus

$$v_\alpha (1 + b_\alpha p_\alpha q_\alpha b_\alpha^{-1}) = 0$$

and $b_\alpha p_\alpha b_\alpha^{-1}$ acts faithfully on V_α.

We now form the module $M = \bigotimes (V_\alpha, v_\alpha)$, which by 2.10 is not (0). The annihilator of $\bigotimes v_\alpha$ contains the set

$$\{1 + b_\alpha p_\alpha q_\alpha b_\alpha^{-1} : \alpha \in \beta\},$$

so that this generates a proper right ideal m in $K[A \wr B]$. But m is comaximal with every non-zero ideal of $F[A^B]$ by construction, and every non-zero ideal of $K[A \wr B]$ intersects $F[A^B]$ in a non-zero ideal by 2.4 and 2.5. Hence m is comaximal with every non-zero ideal of $K[A \wr B]$, proving primitivity.

VARIATION 3.1. We wish to observe that a more complicated version of the construction above yields a faithful, simple module in case K is big. Assume in particular that K is an algebraically closed field with $|K| > |A|\omega$. Let the set X of 3.1 be expanded so that

$$X = F[A^B] - \{0\} \cup P_\omega(B),$$

where $P_\omega(B)$ is the collection of finite subsets of B. Note the X still has cardinality β. Alter the definition of Y_α slightly to include those b for which A_b has received special attention at some stage $\gamma < \alpha$, even if A_b acts trivially. In case α corresponds to an element of $F[A^B]$, we construct V_α in the same way, so suppose α corresponds to the finite set $b_1, \cdots, b_n$ in B. Let W be a fixed simple $K[A^m]$-module besides the trivial one. Such a module must exist for some m, or else the augmentation ideal of $K[A^\omega]$ will be the Jacobson radical. Choose $c_1, \cdots, c_m$ in B so that no b_ic_j lies in Y_α. Then let V_α be the simple module over $K[\Pi A_{b_ic_j}]$ on which $\Pi A_{b_ic_j}$ acts the way A^m does on W, and the other copies of A act trivially. The choice of v_α may be arbitrary. Let $V = \bigotimes(V_\alpha, v_\alpha)$ be the resulting module. By 2.10 and the assumption on K, the module V is simple.

The module M over $K[A \wr B]$ induced from V is the faithful, simple module we desire. It can be explicitly described as follows. For each $b \in B$, let $V(b)$ be a copy of the vector space V, and let

$$M = \bigoplus_{b \in B} V(b).$$

Let B act on M via $v(b)b' = v(bb')$, and identify $V(1)$ with V as a $K[A^B]$-module. The defining relations of the wreath product then yield

$$v(b_1)a_b = v(1)b_1a_b = v(1)a_{b_1b}b_1,$$

so that A_b acts on $V(b_1)$ the way that A_{b_1b} acts on V. The faithfulness of M follows easily. Given any element p_α of $F[A^B]$, we recall that for some b_α, the element $b_\alpha p_\alpha b_\alpha^{-1}$ acts faithfully on $V = V(1)$. Therefore p_α acts faithfully on $V(b_\alpha)$, and M is faithful over $F[A^B]$. This implies faithfulness over $K[A \wr B]$ by 2.4 and 2.5, as before.

For simplicity, it is evident that any non-zero vector in any $V(b)$ is cyclic, since $V(b)$ is simple over $K[A^B]$. Hence it suffices to show that a non-zero vector v

can be sent to a non-zero vector in some $V(b)$. Let $v \neq 0$ lie in

$$V(b_1) \oplus \cdots \oplus V(b_n),$$

and let α be the ordinal corresponding to the set $\{b_1, \cdots, b_n\}$. At stage α we chose $c_1, \cdots, c_m$ in B so that $A_{b_1c_1} \times \cdots \times A_{b_1c_m}$ acts non-trivially on V_α, but $A_{b_ic_j}$ acts trivially on all of V for $i > 1$. This implies that ΠA_{c_j} acts trivially on $V(b_2) \oplus \cdots \oplus V(b_n)$ but non-trivially on $V_\alpha(b_1)$. Choose $a \in \Pi A_{c_j}$ which acts non-trivially on the αth entry of the component of v in $V(b_1)$. Then $v(a-1)$ is a non-zero vector in $V(b_1)$, as desired. We may conclude that M is a faithful, simple $K[A \wr B]$-module.

We now wish to extend 3.1 to subrings of K. This involves adding more V_α's to V which allow some elements of A^B to act on $\bigotimes v_\alpha$ as scalars.

THEOREM 3.2. *Let A and B be two groups with B infinite and $|B| \geqq |A| > 1$. Let K be a field such that $K[A^\omega]$ is semiprimitive, and let S be a subring of K with K generated as S-algebra by $\leqq |B|$ elements. Then $S[A \wr B]$ is primitive in case* (i) *A is not torsion or* (ii) *A is not locally finite and K has characteristic* 0.

PROOF. (i) Let F be the prime subfield of K, let L be the fraction field of S and let a be an element of A of infinite order. We construct a $K[A^B]$-module V as in 3.1, but we add more V_α's. Let

$$X = F[A^B] - \{0\} \cup X_2 \cup X_3,$$

where X_2 is a set of generators of K as L-algebra and X_3 is a set of generators of L over S whose inverses lie in S. By assumption, we can choose X_2 and X_3 so that $|X| = |B| = \beta$. Define Y_α as in 3.1.

If α corresponds to $p_\alpha \in F[A^B]$, we choose an element q_α in $K[A^B]$ such that $1 + q_\alpha p_\alpha$ is not a unit and choose b_α, V_α, and v_α and in 3.1, so that $1 + b_\alpha q_\alpha p_\alpha b_\alpha^{-1}$ annihilates v_α. Alternatively, assume α corresponds to $c_\alpha \in K$, and choose $b_\alpha \in B - Y_\alpha$. Write a_α for the copy of a in A_{b_α}. By 2.7, $a_\alpha - c_\alpha$ is not a unit in $K[A_{b_\alpha}]$, so it lies in a maximal right ideal m_α. Let V_α be the corresponding simple module, with all other copies of A acting trivially, and choose v_α to be the image of 1.

The module $V = \bigotimes (V_\alpha, v_\alpha)$ is non-zero by 2.10, and given $\gamma < \beta$ corresponding to an element c_γ in $X_2 \cup X_3$, we have

$$(\bigotimes v_\alpha) a_\gamma = c_\gamma (\bigotimes v_\alpha).$$

Since $\{c_\gamma\}$ generates K over S, we find that $(\bigotimes v_\alpha) \cdot S[A^B]$ contains $K \cdot (\bigotimes v_\alpha)$.

Consider the element q_γ which arises at a stage γ corresponding to p_γ in $F[A^B]$. The action on $\bigotimes v_\alpha$ of any scalar in $K - L$ involved in q_γ can be copied by some element of $S[A^B]$. Substitute such elements for the corresponding scalars in q_γ to obtain an element $r_\gamma \in L[A^B]$ such that $1 + b_\gamma r_\gamma p_\gamma b_\gamma^{-1}$ annihilates v_γ. We can now conclude as in 3.1 that $\operatorname{ann}(\bigotimes v_\alpha)$ contains the set

$$\{1 + b_\gamma r_\gamma p_\gamma b_\gamma^{-1}\},$$

which must therefore generate a proper right ideal in $L[A \setminus B]$ comaximal with every non-zero ideal of $F[A^B]$. Hence $L[A \setminus B]$ is primitive.

As for $S[A \setminus B]$, its intersection with $\operatorname{ann}(\bigotimes v_\alpha)$ contains the elements obtained by clearing the fractions in $1 + b_\gamma r_\gamma p_\gamma b_\gamma^{-1}$, as well as $c_\gamma^{-1} a_\gamma - 1$, with c_γ^{-1} lying in S. Let $\boldsymbol{m}$ be a maximal right ideal of $S[A \setminus B]$ containing this set. If $\boldsymbol{m}$ contains a non-zero ideal, it must contain a non-zero ideal of S, and since $\boldsymbol{m}$ is maximal, this ideal will be prime. But $\{c_\gamma : \gamma \in X_3\}$ generates L over S, so that any non-zero prime of S must contain some c_γ^{-1}. This forces $\boldsymbol{m}$ to contain 1, a contradiction.

(ii) We now assume that K has characteristic 0 and A contains elements $x_1, \cdots, x_n$ which generate an infinite subgroup. Let $a = x_1 + \cdots + x_n$. The algebra $S[i][A \setminus B]$ is a finite centralizing extension of $S[A \setminus B]$, so that if the first is primitive, so is the second [11, 5.6]. Thus we may assume that S contains i, and write $K = R + Ri$ for some real field R. We construct the spaces V_α as in part (i), except that we define X_2 and X_3 differently. Let X_2 be a set of generators of R over L, all of whose elements are $\geqq n + 1$. To define X_3, choose a set of real elements in S whose inverses generate L over S, and let X_3 consist of all positive integral multiples of these inverses which are $\geqq n + 1$. We can choose these sets so that $|X| = \beta$, and we now construct V just as in part (i), using the different a and 2.9.

We may replace q_α as before with some r_α in $L[A^B]$, and the same argument shows that $L[A \setminus B]$ is primitive. To deal with $S[A \setminus B]$, let γ be an ordinal corresponding to c_γ in X_3, and write c_γ as $k_\gamma s_\gamma^{-1}$ for some integer k_γ and $s_\gamma \in S$. Then let $\boldsymbol{m}$ be a maximal right idealof $S[A \setminus B]$ containing its intersection with $\operatorname{ann}(\bigotimes v_\alpha)$. In particular, $\boldsymbol{m}$ contains the elements

$$\{s_\gamma a_\gamma - k_\gamma\}.$$

The same argument as in (i) shows that if $\boldsymbol{m}$ contains any non-zero ideal, it contains a non-zero prime of S. But every non-zero prime must contain some s_γ, and we can find $\delta \neq \gamma$ with $s_\gamma = s_\delta$ and k_δ relatively prime to k_γ. This forces $\boldsymbol{m}$ to contain 1, a contradiction.

REMARKS. (1) As in variation 3.1, we can adjust the construction of 3.2 to produce a faithful, simple module for $S[A \wr B]$ in case K is algebraically closed and $|K| > |A|\omega$.

(2) We note one consequence of 3.2. Let A be an infinite, finitely-generated torsion group. Then $G = A \wr A$ is a finitely-generated, torsion group with the property that $S[G]$ is primitive for S any field or countable domain of characteristic 0.

The results on semiprimitivity described in the introduction can now be obtained easily.

COROLLARY 3.3. *Let K be a field and A a group such that $K[A^\omega]$ is semiprimitive. Assume* (i) *A is not torsion or* (ii) *A is not locally finite and K has characteristic* 0. *Then for any field F of the same characteristic, $F[A^\omega]$ is semiprimitive.*

PROOF. By 2.3, $K[A^\omega]$ is separable, so it remains semiprimitive under base field extension. Thus we may assume that $K \supset F$. Let $|A|[K:F] = \lambda$ and let B be a locally finite group with $|B| = \lambda$. Then by 3.1, the ring $F[A \wr B]$ is primitive, and $F[A^\lambda]$ is semiprimitive by 2.6. We claim that this implies $F[A^\omega]$ is semiprimitive.

Let p be in the radical of $F[A^\omega]$, and choose a finite number of copies of A in A^λ to correspond to the copies in A^ω containing the support of p. Let p' be the corresponding element in $F[A^\lambda]$. For any q' in $F[A^\lambda]$, the support of $p'q'$ is also a finite number of copies of A, and we can choose an analogous element q in $F[A^\omega]$. Then $1 + pq$ has an inverse by assumption, and the corresponding element in $F[A^\lambda]$ is an inverse to $1 + p'q'$. Therefore p' is in the radical of $F[A^\lambda]$, forcing $p' = 0 = p$.

COROLLARY 3.4. *If F has characteristic* 0, *then $F[A^\omega]$ is semiprimitive for any group A. If A has no element of order p and F has characteristic p, then $F[A^\omega]$ is semiprimitive in case A is locally finite or A is not torsion.*

PROOF. The case that A is locally finite is an easy consequence of Maschke's Theorem. In the other cases, semiprimitivity is known for fields which are not algebraic over the prime subfield [10, 7.3.13–14], and we can apply 3.3 to the remaining cases.

REMARK. Passman has provided a direct proof that $F[A^\omega]$ is semiprimitive in case F has characteristic 0 and A is not torsion.

We close this section with some questions raised by these results:

QUESTION 1. In case A is locally finite, what can be said about the primitivity of $S[A \wr B]$ for S a domain?

QUESTION 2. In characteristic p, if A is torsion but not locally finite, does the primitivity of $K[A \wr B]$ descend to subrings of K of a suitable size?

QUESTION 3. If $|A| > |B|$ and B is infinite, what can be said about the primitivity of $K[A \wr B]$?

4. A primitivity result in characteristic p

The semiprimitivity assumption of section 3 is not always necessary. Passman has proved that for a finite p-group P over a field K of characteristic p, the algebra $K[P \wr \mathbb{Z}]$ is primitive. In this section we describe an explicit faithful, simple module for this algebra, as part of a more general result. Recall Kolchin's theorem, which states that a multiplicative semigroup of unipotent matrices may be simultaneously put in triangular form, so that it has a simultaneous eigenvector with eigenvalue 1 [8, p. 100]. This implies the following presumably well-known fact:

PROPOSITION 4.1. *Let P be a finite p-group and let K be a field of characteristic p. Then any simultaneous P-eigenvector in the regular representation of $K[P]$ is a scalar multiple of $w = \Sigma_{x \in P} x$, and every non-zero submodule contains w.*

PROOF. Let M be a non-zero submodule of $K[P]$. Since P is a p-group, every element must map to a unipotent matrix in the representation on M. Therefore, by Kolchin's theorem, M contains a simultaneous P-eigenvector. Any non-zero eigenvector in $K[P]$ must have P as its support, and since every element of P acts with eigenvalue 1, the coefficients must all be the same. Thus the eigenvector is a multiple of w.

Our main result in this section is

THEOREM 4.2. *Let G be a group containing a series of subgroups*

$$\cdots \subset G_{-1} \subset G_0 \subset G_1 \subset \cdots$$

such that $\bigcap G_i = 1$, $\bigcup G_i = G$, each G_i is normal in G and G_{i+1}/G_i is isomorphic to a fixed finite p-group P for all i. Assume that there is an automorphism φ of G with $\varphi(G_i) = G_{i+1}$. Let H be the semi-direct product of G and $\mathbb{Z} = \langle x \rangle$, with

$x^{-1}gx = \varphi(g)$ for all $g \in G$. Then $K[H]$ is primitive for any field K of characteristic p.

PROOF. We will construct a faithful, simple $K[H]$-module. Let V_i be the K-vector space $K[G/G_i]$, and let $V = \bigoplus V_i$. We give V a $K[H]$-module structure as follows: Let G/G_i act on V_i via the regular representation and define the G-action on V to be the naturally induced one. Thus, for $g' \in G$ and $G_ig \in V_i$, we have

$$(G_ig)\cdot g' = G_i(gg').$$

The map φ induces an isomorphism $G/G_i \to G/G_{i+1}$, which we use to define the action of x:

$$(G_ig)\cdot x = G_{i+1}\varphi(g).$$

This is well-defined since

$$(G_ig)\cdot x^{-1}g'x = (G_{i-1}\varphi^{-1}(g))\cdot g'x = G_ig\varphi(g').$$

The resulting module will be faithful, but not simple; we must pass to a homomorphic image.

For $j > i$, the group G_j/G_i is a p-group which embeds in G/G_i. Let w_i^j in V_i be the sum of the cosets G_j/G_i. Also let w_i^i be the coset G_i in V_i. By 4.1, any non-zero $K[G/G_i]$-submodule of V_i must contain one of the elements w_i^j. Notice that for any transversal T of G_{j+1}/G_j, we have

$$\sum_{g\in T} w_i^j\cdot g = w_i^{j+1}. \tag{1}$$

Since $\varphi(G_i) = G_{i+1}$, we find that

$$w_i^j\cdot x = w_{i+1}^{j+1}$$

for $j \geqq i$. Thus the following set of vectors is x-invariant:

$$\{w_i^{i+1} - w_{i+1}^{i+1} : i \in \mathbb{Z}\}.$$

Let W be the $K[H]$-submodule of V generated by this set. Then W is spanned by the images of this set under G, so that by (1) W must include

$$\{w_i^j - w_{i+1}^j : i < j\}.$$

Define $\bar{V}$ to be the $K[H]$-module V/W. We claim that $\bar{V}$ is faithful and simple. Observe that the vectors w_i^i are cyclic in V, since each is cyclic in V_i and x acts transitively on them. But W contains $w_i^j - w_{i+1}^j$ for any $i < j$, so the image of each w_i^i must be cyclic in $\bar{V}$. As we noted above, by 4.1, any non-zero $K[G]$-submodule of V_i contains some w_i^j. Thus the image of any non-zero element of some V_i is cyclic in $\bar{V}$, and simplicity will be proved if we show, for any $\bar{v} \neq 0$ in $\bar{V}$, that $\bar{v} \cdot K[H]$ contains such an element.

Let $v \in V$ be a pre-image of $\bar{v}$ and write $v = v_m + \cdots + v_n$ with $v_i \in V_i$ and $m < n$. We may assume that v_m is non-zero, and is not equivalent to a vector in $\bigoplus_{i>m} V_i$ modulo W. Then v_m cannot be an eigenvector for G_{m+1} since the G_{m+1}-eigenvectors of V_m are precisely the elements in $w_m^{m+1} \cdot K[G]$, which are equivalent to elements of V_{m+1} modulo W. (To see this, just note that any subset of G/G_m invariant under multiplication by G_{m+1} is a union of G_{m+1}-cosets.) On the other hand, G_{m+1} acts trivially on V_i for $i > m$. Therefore, we can choose $g \in G_{m+1}$ such that $v_m \cdot g \neq v_m$, and we obtain

$$v \cdot g - v = v_m \cdot g - v_m,$$

a non-zero element of V_m. By the preceding remarks, $\bar{v}g - \bar{v}$ is cyclic, and $\bar{V}$ is simple.

For faithfulness, let $\Sigma x^i p_i$ annihilate $\bar{V}$, with $p_i \in K[G]$. Then

$$0 = \bar{w}_j^j \left(\sum_i x^i p_i \right) = \sum_i \bar{w}_{j+i}^{j+i} \cdot p_i,$$

so each p_i annihilates all the elements $\{\bar{w}_k^k\}$. This means that in V, we have $w_k^k p_i \in W$. But this element is also in V_k, and $W \cap V_k = (0)$, so p_i annihilates all the vectors w_k^k. Choose k small enough that the support of p_i lies outside G_k. Then w_k^k represents the identity element in the regular representation of $K[G/G_k]$ and p_i annihilates w_k^k, so $p_i = 0$.

Corollary 4.3. *Let P be a finite p-group and let K be a field of characteristic p. Then $K[P \wr \mathbb{Z}]$ is primitive.*

Proof. Let G be the restricted direct product of the groups $\{P_i : i \in \mathbb{Z}\}$, where each P_i is a copy of P, and let

$$G_i = \prod_{j \leq i} P_j.$$

Define $\varphi : G \to G$ to be the automorphism which sends P_i identically to P_{i+1} for all i. Then $P \wr \mathbb{Z}$ is the semidirect product of G by $\mathbb{Z}$ with respect to φ, and the conditions of 4.2 are satisfied.

Remarks. (1) The proof of 4.2 provides an explicit faithful, simple module for $K[P \wr \mathbb{Z}]$ in 4.3. The construction of 4.2 can be modified to show that, more generally, $K[P \setminus Q]$ is primitive for P a p-group, K a field of characteristic p, and Q a group which is not torsion.

(2) It would seem likely that $K[G \setminus \mathbb{Z}]$ is primitive for any finite group G.

(3) We apply 4.2 to another family of group rings in the next section.

We thank M. Lorenz for the suggestion that the results of 4.3 and 5.2 could be combined in a statement like 4.2.

5. Primitivity of another family of groups

In this section we prove that for any integer $r > 1$, the group presented by $\langle x, y : x^{-1}yx = y^r \rangle$ has a primitive group ring over any field. For most fields the proof is quite easy, but in certain cases we use 4.2. To see just how easy the proof can be in optimal circumstances, let us first show the following. Recall that an absolute field is one which is algebraic over a finite field.

PROPOSITION 5.1. *Let K be an algebraically closed field which is not absolute and let $G = \langle x, y : x^{-1}yx = y^r \rangle$ for $r > 1$. Then $K[G]$ is primitive.*

PROOF. Let V be the K-vector space with basis $\{v_n : n \in \mathbb{Z}\}$ and choose an element $c \in K$ which is not a root of unity or 0. Choose a sequence of scalars $c_i \in K$ so that

$$c_i^r = c_{i+1}, \qquad c_0 = c.$$

Since c is not a root of unity, these elements are all distinct. We make V a $K[G]$-module by defining

$$v_n \cdot x = v_{n+1},$$

$$v_n \cdot y = c_n v_n.$$

This is well-defined, since $v_n \cdot x^{-1}yx = c_{n-1}v_n = c_n^r v_n = v_n \cdot y^r$.

For any $v \neq 0$ in V, the space $v \cdot K[y]$ contains a basis vector. To see this, let

$$v = \sum_{i=m}^{n} a_i v_i$$

with $m < n$ and $a_m a_n \neq 0$. Then

$$v(y - c_n) = \sum_{i=m}^{n} (c_i - c_n) \cdot v_i$$

is non-zero and involves fewer basis vectors. The action of x on the basis is transitive, so v is cyclic and V is simple.

For faithfulness, let $p = \Sigma x^i p_i(y)$ annihilate V. Multiplying on the right by a suitable power of y, we may assume that each $p_i(y)$ is in $K[y]$. Then

$$0 = v_n \cdot p = \sum_i v_{i+n} \cdot p_i(y),$$

so that each $p_i(y)$ must annihilate V. But $v_n \cdot p_i(y) = p_i(c_n)v_n$, so that p_i vanishes on an infinite set of scalars. Therefore $p_i = 0$ and $p = 0$.

The general proof is an elaboration of this, except that it does not work when $r = p$ and K is absolute of characteristic p. But Theorem 4.2 applies in case $r = p$ and K has characteristic p. Thus, we can prove

THEOREM 5.2. *Let F a field and let $G = \langle x, y : x^{-1}yx = y^r \rangle$ for r an integer $>$ 1. Then $F[G]$ is primitive.*

PROOF. (i) F is not absolute.

Choose an element $c \neq 0$ in F which is not a root of unity, and let K be the subfield of $\bar{F}$ generated over F by a sequence of scalars $\{c_i\}$ chosen as in 5.1, with $c_i^r = c_{i-1}$ and $c_0 = c$. Construct the $K[G]$-module V exactly as in 5.1. The same proof shows that V is faithful over $F[G]$. For any $v \neq 0$ in V, we can again show that $v \cdot F[G]$ contains a basis vector, but the proof must be modified. The problem is that the elements c_i do not all necessarily lie in F. We choose N so that the basis vectors involved in vx^N include v_0. Then $vx^N(y - c_0)$ lies in $v \cdot F[G]$ and involves one less basis vector. Thus we obtain a non-zero multiple of a basis vector in $v \cdot F[G]$. It remains to show that

$$v_0 \cdot F[G] \supseteq K \cdot v_0.$$

But $v_0 x^{-n} y x^n = c_{-n} v_0$, so $v_0 \cdot F[G]$ contains all the elements $c_n \cdot v_0$, as desired.

(ii) F is absolute of characteristic $\neq r$.

Let $c_{-n} = 1$ for $n \geqq 0$ and let c_n be a primitive (r^n)th root of unity in $\bar{F}$, chosen so that $c_n^r = c_{n-1}$. The sequence $c_1, c_2, \cdots$ consists of distinct scalars. Let K be the subfield of $\bar{F}$ generated over F by this sequence, and let V be the $K[G]$-module constructed as in 5.1. It is still faithful over $F[G]$, and simplicity will follow as before if we show that for any $v \neq 0$, the module $v \cdot F[G]$ contains $K \cdot v_0$. Applying a suitable power of x if necessary, we may assume that

$$v = a_0 v_0 + \cdots + a_n v_n$$

with $n > 0$ and $a_0 a_n \neq 0$. Then

$$v(y - x^{-1}yx) = \sum_{i=0}^{n} (c_i - c_{i-1}) a_i v_i,$$

and $c_0 = c_{-1}$ while $c_i \neq c_{i-1}$ for $i > 0$. Thus this vector is non-zero in $v \cdot F[G]$ and involves fewer basis vectors. In this way we obtain a non-zero multiple of v_0 in $v \cdot F[G]$, and the argument is completed as in (i).

(iii) F has characteristic p and $r = p$.

We claim that G can be presented as a semi-direct product having the form described in Theorem 4.2. Let

$$z_n = x^{-n} y x^n$$

for any integer n. Then the elements $\{z_n\}$ generate an abelian subgroup A of G, with

$$z_n^p = z_{n+1}.$$

Let A_n be the subgroup of A generated by $\{z_{-n}, z_{-n+1}, \cdots\}$. Then $A_n \subset A_{n+1}$ and A_{n+1}/A_n is cyclic of order p. We can define an isomorphism φ of A by $\varphi(z_n) = z_{n-1}$, and φ sends A_n to A_{n+1}. This isomorphism coincides with conjugation by x:

$$x z_n x^{-1} = z_{n-1} = \varphi(z_n).$$

Thus G is the semi-direct product of A by $\mathbb{Z}$ with respect to φ, and 4.2 applies.

6. Free products of finite groups

One of the early results on primitive group rings is the theorem of Formanek [3] that if G and H are two non-trivial groups, one of which has more than two elements, then $F[G * H]$ is primitive for any field F. In addition, given $|G| \geqq |H|$, he proved that $R[G * H]$ is primitive for any domain R with $|R| \leqq |G|$. In [5], we described a construction of faithful, simple modules for such group rings, assuming G is infinite and H is residually $< |G|$. We wish in this section to construct faithful, simple modules in case G and H are certain finite groups. This procedure can presumably be extended to any two finite groups, but the construction would be somewhat more complicated to describe.

THEOREM 6.1. *Let G be a finite group of size $n > 2$, let $H = \langle y : y^2 = 1 \rangle$, and let R be a countable domain or a field. Then $R[G * H]$ is primitive.*

PROOF. Let K be the smallest infinite field containing R. For each integer i, let V_i be a K-vector space isomorphic to $K[G]$ with basis $v_i^1, \cdots, v_i^n$, and let $V = \bigoplus V_i$. We make V a $K[G]$-module by letting G act on each V_i via the regular representation. The action of y will be defined to make V faithful and simple over $R[G * H]$.

First let

$$v_i^n \cdot y = v_{i+1}^1,$$

so that y switches the last basis vector of one V_i with the first basis vector of the next V_{i+1}. This insures that $G * H$ acts transitively on the basis of V. For any $1 < j < n$, we will define the y-action on v_i^j to send it to a scalar multiple of some v_k^j, with $i \neq k$, preserving the superscript. For convenience, let us say that v_i^j has type j. Thus y switches the type of vectors of type 1 and n, while preserving all other types.

For non-negative subscripts, and $1 < j < n$, let

$$v_{2i}^j \cdot y = c_i v_{2i+1}^j,$$

$$v_{2i+1}^j \cdot y = c_i^{-1} v_{2i}^j,$$

where $\{c_i : i \geqq 0\}$ is a distinct set of non-zero elements of K which span K over R, and $c_0 = 1$. We claim that, however y is defined on the rest of V, within the indicated restrictions, the resulting module will be simple over $R[G * H]$.

Any non-zero multiple of a basis vector is cyclic, since x and y act transitively on the basis and their action can be used to multiply a basis vector by any c_i. Thus it suffices to prove that for any $v \neq 0$ in V, the module $v \cdot R[G * H]$ contains a non-zero multiple of v_0^1.

Assume first that v involves only type 1 vectors. Let $g, h \in G$ be the elements for which

$$v_i^1 \cdot g = v_i^n, \qquad v_i^1 \cdot h = v_i^2$$

for all i, and define $z = gyhyh^{-1}$. Then gy sends v_i^1 to v_{i+1}^1 and for $i \geqq 0$,

$$v_{2i}^1 \cdot z = c_i^{-1} v_{2i}^1,$$

$$v_{2i+1}^1 \cdot z = c_{i+1} v_{2i+3}^1.$$

Thus type 1 vectors with even subscript are multiplied by scalars and those with odd subscript get pushed to the next odd component. We can apply a suitable power of gy to v so that

$$v = a_0 v_0^1 + \cdots + a_m v_m^1$$

and $a_0 \neq 0$. Suppose some $a_{2n} \neq 0$ with $n > 0$. Since $c_0 = 1$,

$$v(z-1) = \sum (c_i^{-1} - 1) a_{2i} v_{2i}^1 + (\text{linear combination of odd type 1 vectors})$$

involves fewer even type 1 vectors. Proceeding in this way, we can eliminate all the even type 1 vectors. Applying gy to the result produces a new linear combination of even type 1 vectors, and we eventually obtain a non-zero multiple of v_0^1 in $v \cdot R[G * H]$.

Since any element of $G * H$ permutes types, there is an integer $k > 0$ such that z^k preserves all types. As a result, given $v \neq 0$ in V, we may assume that it involves type 1 vectors and use z^k and the same procedure as above to eliminate them all, while preserving all the other types involved. We can thereby reduce v to a non-zero vector of a single type, proving that v is cyclic.

To prove faithfulness, it suffices to check that for any finite set of words in $G * H$, there is a basis vector which each word sends to a distinct vector. The collection of finite sets of words in $G * H$ is countable, and for each element in it we define the y-action on finitely many of the components V_i so that the given set of words satisfies the faithfulness condition. The procedure is as follows: Given $w_1, \cdots, w_m$, choose N much smaller than any i for which the y-action has been defined on V_i, and select a basis vector v_N^j with $1 < j < n$. We proceed inductively, defining a new y-action each time y arises in one of the words. For instance, let $w_1 = uyu'yu''$ and suppose we have defined $v_N^j u$ as some basis vector with much lower subscript, chosen so that $v_N^j uyu'$ will be a basis vector which has not arisen yet either. Continue in this way with each word in turn. Of course, if we meet a vector of type 1 or n, the y-action is already determined and we proceed to the next occurrence of a y. After using all the words $w_1, \cdots, w_m$, if some V_i was used but the y action was not defined on all of V_i, complete the definition arbitrarily, provided the appropriate types are preserved. The result is that V is faithful.

Remarks. (1) The procedure above generalizes easily to any finite cyclic group H. For arbitrary finite groups H, we can presumably construct a module in which any $h \in H$ permutes basis vectors in cycles corresponding to the regular representation on H, but the details would be rather messy. A procedure of this type was used in the module construction for free products in [5], where a faithfulness argument analogous to the one above is described in more detail.

(2) It seems plausible that a construction of the above sort could be used for free products or products with amalgamation of certain pairs of finite-dimensional algebras. For instance, we could have modified the construction to allow $K[G]$ to be any finite-dimensional algebra which acts transitively on its basis. This permits the construction of faithful, simple modules for group rings of certain amalgamated products. The primitivity of such group rings also follows by modifying Formanek's original proof, as K. Brown has observed [1]. Of

course, it is known by a result of Lichtman [9] that the free product of any two algebras over a field is primitive, provided the dimensions are at least 2 and 3. We wonder to what extent the constructions above and in [5] can be used to obtain faithful, simple modules for such free products.

Note added in proof. A modification of the proof of Theorem 3.2 can be made to prove that $S[A \setminus B]$ is primitive as long as $K[A]$ is not algebraic. This provides a uniform proof for cases (i) and (ii) in the Theorem, but, as noted in the Introduction, it is unknown whether the resulting theorem applies to a wider class of groups A.

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